Current Algebras and Symmetries in Bootstrap Theory*†

R. F. DASHEN AND S. C. FRAUTSCHI‡
California Institute of Technology, Pasadena, California
(Received 8 November 1965)

In the first paper of this series we showed how, in the bootstrap theory, the currents associated with the hadrons could be determined from a set of self-consistency conditions. In the present paper we show that these "self-consistent" currents satisfy a current algebra. The proof is accomplished without recourse to any approximate model. It includes the interesting case of nonconserved currents. The convergence of sum rules derived from current algebras is investigated in detail, and shown to be most rapid when no "nonbootstrap" terms are present. Using these convergence properties, we discuss how and when current algebras can give rise to hadron symmetries.

I. INTRODUCTION

IT has been repeatedly stressed by Gell-Mann[1] that the equal-time commutation relations of currents can be very helpful in understanding the properties of hadrons. These commutation relations are supposed to be exact, holding even if the currents are not conserved.

The idea that currents, even when they are not conserved, may satisfy *exact* equal-time commutation relations has usually been taken from special field-theoretical models. The emergence of such simple, precise relations from the complicated coupled equations of the bootstrap might at first appear like an impossible accident. We have found, however, that the bootstrap *does* produce these equal-time commutation relations.

The study of currents in the bootstrap theory was initiated in the first paper of this series,[2] where we showed how the currents associated with the hadrons could be determined from a set of bootstrap-like consistency conditions. The present paper contains the demonstration that these "bootstrap" currents satisfy a current algebra, even when the currents are not conserved.

In addition to the intrinsic interest of this connection between the bootstrap and algebraic approaches, it helps complete the program of specifying the properties of currents in the bootstrap approach. The first paper of this series started the program with a linear study of currents, including some approximate calculations which indicated that the approach could work in practice. The linear study was clearly incomplete, however; for one thing, it did not give us any way to specify the normalization of the currents. The algebraic properties derived in the present paper partially fill this need.[1,3]

We should specify carefully in what sense a proof of the algebraic properties has been achieved. To begin with, we are basing our argument on the assumption that a consistent bootstrap theory of strong interactions exists. Even given the existence of a bootstrap theory, our proof is not rigorous in the manner of a proof from axiomatic field theory. The reason for this is, of course, that a concise and complete set of axioms for the bootstrap theory does not yet exist. We have, however, based our arguments on general principles which we expect will hold in whatever form the bootstrap might take. That is, we do not employ any approximate models in the proof.

Another topic, discussed in the later sections of the present paper, is the relation of current algebras to strong-interaction symmetries. Empirically, there appears to be a very close correlation between the two. The reason for this correlation has generally appeared to be a lucky but mysterious accident. Some light has been shed on this question by Lee,[4] and Dashen and Gell-Mann,[5] who have shown how current algebras could give rise to hadron symmetries provided that the currents have special properties such as rapid convergence of their commutators. One of our goals in this paper is to show how and when these special properties can emerge from the bootstrap.

During the course of our discussion of symmetries, we investigate in detail the convergence of sum rules derived from current algebras. These results are likely to be of independent interest.

The organization and main points of this paper can be summarized as follows. In Sec. II, the treatment of currents given in the first paper of this series is reviewed, processes of second order in the currents are introduced, and the asymptotic properties of amplitudes associated with the currents are summarized. In Sec. III we show, using purely physical arguments, that the currents derived from a bootstrap should form an algebra. The following section contains a more mathematical proof of the same assertion, in which we use dispersion relations and S-matrix theory.

* Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract AT(11-1)-68 for the San Francisco Operations Office, U. S. Atomic Energy Commission.

† A preliminary report on some of the work presented here is contained in the *Proceedings of the 1965 Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, California, 1965).

‡ Alfred P. Sloan Foundation Fellow.

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964). See also Refs. 5, 10, and 22.

[2] R. Dashen and S. Frautschi (to be published). We refer to this paper as (I).

[3] An example of the use of commutation relations to determine the normalization of a current is the Adler-Weisberger calculation of the beta-decay ratio C_A/C_V [S. Adler, Phys. Rev. Letters **14**, 1051 (1964); and W. Weisberger, *ibid.* **14**, 1047 (1965)].

[4] B. Lee, Phys. Rev. Letters **14**, 676 (1965).

[5] R. Dashen and M. Gell-Mann, Phys. Letters **17**, 142 (1965).

It is perhaps worth stating now that Sec. IV contains a number of formal manipulations of dispersion relations, none of which are justified with strict mathematical rigor. The physical arguments of Sec. III lead to the same conclusions, however, without relying on this sort of formal mathematics. In view of this fact many readers may wish to initially bypass Sec. IV, up to Eq. (18) where we turn from time components of currents to space components.

Section V contains a discussion of the convergence properties of sum rules. We find that the number of intermediate states necessary to saturate the sum rule is highly dependent on the type of current under consideration. In Sec. VI we investigate in detail how current algebras can lead to supermultiplets; the convergence properties of commutators as discussed in Sec. V. are important here. In the final section we discuss specifically how $SU(3)$ and $SU(6)$ could be fit into the framework developed in this paper.

II. SELF-CONSISTENT CURRENTS

In (I) we showed how, if there are no elementary hadrons, the currents associated with the strongly interacting particles could be determined by a set of bootstrap-like self-consistency conditions. In this section we review the main results of (I) and take up a few further aspects of the general idea of a bootstrap approach to currents which will be needed in our later work.

The basic idea behind the bootstrap approach to the hadrons can be expressed as follows. We assume that there exists some dynamical scheme in which, given a physical parameter such as a mass or a coupling constant, one can calculate this quantity in terms of input data composed of other physical quantities. When all quantities have been calculated in this way, the input and output are required to match. The hope is then that the resulting self-consistency equations have a reasonably small number of solutions, one of which corresponds to the observed hadrons.

Now the bootstrap idea is usually thought of as pertaining to the properties of hadrons which involve only the strong interactions. There is no reason, however, why it should not also apply to the weak and electromagnetic parameters of the hadrons. Indeed, if there are no elementary hadrons, the bootstrap would seem to provide the only means of determining these parameters.

To see how the bootstrap scheme would work for the electromagnetic properties of hadrons, consider the following chain of thought. Suppose that we want to calculate the magnetic moment of the deuteron. Everyone believes that the deuteron is a composite object composed mostly of one neutron and one proton. Thus we should be able to calculate its magnetic moment in terms of the nucleon magnetic moments. Going another step up the ladder, consider a calculation of the nucleon magnetic moments. Nowadays, hardly anyone believes that the nucleon is elementary; it is almost certainly a bound state consisting to a considerable extent of one pion and one nucleon. From this point of view, we should be able to calculate the nucleon magnetic moments in terms of input which would include, among other things, the nucleon moments themselves. Thus we see the beginning of a series of self-consistency conditions, which in a bootstrap theory would determine the nature of the electromagnetic interaction of the hadrons.

In general, a bootstrap approach to the electromagnetic interaction of the hadrons would proceed as follows. The electromagnetic properties of the hadrons are specified by the matrix elements $\langle b|J^\nu(0)|a\rangle$ of the electric current where a and b are arbitrary systems of hadrons. In the bootstrap, if we are given some matrix element $\langle b|J^\nu(0)a\rangle$, we are supposed to be able to calculate it in terms of other matrix elements $\langle d|J^\nu(0)|c\rangle$ of the current. Assuming that it is meaningful to work to first order in electromagnetism, we would arrive at an infinite set of linear, homogeneous equations relating the matrix elements of J. Schematically, we can write these equations as

$$J=X[J], \tag{1}$$

where X is a homogeneous linear operator which is determined by the strong interactions. Evidently, the electromagnetic current would have to correspond to an "eigenvector" of X whose eigenvalue is exactly one.

The weak currents should also satisfy (1). No doubt there are other solutions which do not correspond to observed interactions.[6] We call any solution to (1) a self-consistent current. Clearly, in a bootstrap theory the strong interactions specify a well-defined class of self-consistent currents.

Mathematically, one can formulate the ideas expressed above in terms of S-matrix theory. We have carried this out in (I) and will summarize the main ideas and results below.

In dealing with a current J in S-matrix theory, it is convenient to imagine a particle called, for no particular reason, θ. We then denote by $J_{ba}(q^2)$ the amplitude for the process $a\rightarrow b+\theta$, where $(q^2)^{1/2}=[p_a-p_b)^2]^{1/2}$ is the "mass" of θ; J_{ba} is equal, apart from kinematical factors, to $\langle b|J(0)|a\rangle$. In S-matrix theory, the linear self-consistency equations for J are obtained by writing dispersion relations for $J_{ba}(q^2)$. In order that the equations be homogeneous, there must be no undetermined subtractions in the dispersion relations for the J's. Appendix A of (I) contains a discussion of the subtraction question. It is concluded there that if (i) there are no subtractions in the strong interactions, i.e., if a strong interaction bootstrap makes sense, and (ii) if all form factors tend to zero for large q^2, which one would rather expect if all hadrons are composite, then there

[6] In the next section we show that the solutions to (1) form a Lie algebra. From this fact one can conclude that (1) must have solutions other than the known weak and electromagnetic currents.

is no foreseeable reason why there should be undetermined subtractions in the dispersion relations for the J's.

The dispersion relations for the K_{ba}'s fall into two general classes:

(*i*) *Single-variable dispersion relations in* q^2. These are the usual equations for form factors which have the property of determining the J's for all q^2 once all the matrix elements $J_{ba}(q^2)$ for some fixed $q_0{}^2$ are known. These equations do *not* place any restriction on the $J_{ba}(q_0{}^2)$.

(*ii*) *Dispersion relations for fixed* q^2. Here we fix q^2 at, say, $q_0{}^2$ and considering $J_{ba}(q_0{}^2)$ as the amplitude for $a\to b+\theta$, write the "mass shell" dispersion relations for this amplitude just as we would for a strong interaction amplitude. Writing all the fixed q^2 dispersion relations leads to a set of equations which have the form

$$J_{ba}(q_0{}^2)=\sum_{cd} X_{ba,dc}(q_0{}^2)J_{dc}(q_0{}^2)\,, \tag{2}$$

where the matrix X is completely determined by the strong interactions.

Evidently, the single-variable dispersion relations (i) are not sufficiently restrictive to serve as a basis for a "bootstrap" theory of the currents. Instead, Eq. (2) which follows from the fixed q^2 dispersion relations is the key.

For each fixed $q_0{}^2$, the self-consistent currents must satisfy the homogeneous X-matrix equation (2). In the input-output formalism of the bootstrap, the left-hand side of (2) is to be thought of as the output amplitude for which a dispersion relation has been written, and the J's appearing on the right are the input amplitudes which appear, through the unitarity condition, inside the dispersion integral.

To test the idea that the currents are, in fact, determined by an equation like (2), we have carried out some model calculations. Specifically, we looked at the usual static model of baryons and resonances and asked what currents would be self-consistent in this model. It turns out that all the observed weak and electromagnetic currents are, in this approximation, self-consistent. As a by-product we obtained a number of predictions for ratios among quantities like magnetic moments, all of which are in agreement with experiment. The detailed results of these calculations, and further properties of Eq. (2), are given in I.

We have already mentioned that the amplitudes $J_{ba}(q^2)$ for $a\to b+\theta$ can satisfy, for fixed q^2, unsubtracted dispersion relations. As shown in Appendix A of (I) they have, in fact, the same asymptotic behavior as a purely strong-interaction amplitude. This conclusion, which will be very important in our future discussions about current algebras, is nontrivial and would not necessarily hold in a nonbootstrap approach to the currents. We note, in this connection, that the amplitude J_{ba} is defined to be linear in the θ-hadron coupling and therefore is not subject to any obvious unitarity bound.

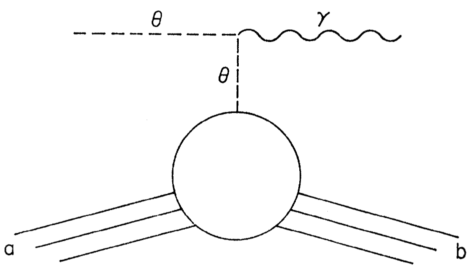

FIG. 1. A nonbootstrap term in the amplitude for $a+\theta\to b+\gamma$.

We now take up two topics which were not discussed in (I) and which will be used heavily in the present paper. The first concerns asymptotic behavior of various amplitudes relating to nonconserved vector and axial currents, and the second concerns the properties of second-order processes (in weak and electromagnetic interactions), like Compton scattering.

From our point of view, a nonconserved vector (axial) current consists of two distinct pieces; a scalar (pseudoscalar) part $J_0=\partial_\alpha J^\alpha$ and a transverse part $J_1{}^\nu=J^\nu-(\partial_\nu\partial_\mu/\Box^2)J^\mu$. As pointed out in (I), J_0 and J_1 satisfy, as long as $q^2\neq 0$, separate X-matrix equations. In a bootstrap, then, the amplitudes $J_{1ba}{}^\nu$ and J_{0ba} are directly determined by the dynamics and $J_{ba}{}^\nu$ is *defined* by[7]

$$J_{ba}{}^\nu=J_{1ba}{}^\nu-i(q^\nu J_{0ba}/q^2)\,. \tag{3}$$

A consequence of this is that $J_{1ba}{}^\nu$ and J_{0ba} will obey unsubtracted dispersion relations and should have, for fixed q^2, the same asymptotic behavior as a strong interaction amplitude. From (3) it is evident that $J_{ba}{}^\nu$ will have a worse asymptotic behavior since q^ν/q^2 can become large. This should be contrasted with the common assumption that $J_{0ba}=q^\nu J_{ba}{}^\nu$ will have an asymptotic behavior worse than that of $J_{ba}{}^\nu$.

We can easily extend our formalism to include second-order processes like Compton scattering. In general, however, the complete second-order amplitudes will contain some nonbootstrap terms. To see this, consider the process $a+\theta\to b+\gamma$, where θ is charged and γ is a photon. The complete amplitude will contain the "θ exchange" graph shown in Fig. 1. Since we are treating θ as an elementary particle, this diagram will cause a subtraction to appear in the dispersion relations, with the result that the $\theta\theta\gamma$ coupling cannot be determined by dispersion relations alone. We can, however, treat the amplitude for $a+\theta\to b+\gamma$, less the θ exchange diagram, by bootstrap means. Later, if we want the full amplitude, the θ exchange diagram can always be added back in explicitly. In general, it is clear that nonbootstrap terms in the second-order amplitude

[7] There is a relation between J_1 and J_0 because as $q^2\to 0$, J^ν must remain finite. A detailed discussion of this point will be given in the third paper of this series.

come from a well-defined class of diagrams and can always be removed, leaving a piece of the amplitude which is amenable to the bootstrap. To this end, we define second-order amplitudes $A_{ba}{}^{\theta_2\theta_1}$ as that part of the amplitude for $a+\theta_1 \rightarrow b+\theta_2$ which contains only hadrons as internal particles. The same arguments which allowed us to conclude that the amplitudes $J_{ba}{}^1$ and $J_{ba}{}^2$ satisfy, for fixed q^2, unsubtracted dispersion relations, also allows us to conclude that $A_{ba}{}^{\theta_2\theta_1}$ is free of undetermined subtractions. In particular, the amplitudes $A_{ba}{}^{\theta_2\theta_1}$ should have the same asymptotic behavior as strong interaction amplitudes. Again this is a nontrivial conclusion, since there are no elastic intermediate states in A and, hence, no obvious unitarity bound.

In the following sections, we will often be working with second-order processes involving nonconserved currents. It is particularly important to pin down, in the case of nonconserved currents, just which amplitudes will have the subtraction-free asymptotic behavior of strong-interaction amplitudes. To this end, we consider two nonconserved currents $J_1{}^\nu$ and $J_2{}^\nu$.

Let θ_1 and θ_2 be spin-one particles coupled to the transverse parts of J_1 and J_2, and ϕ_1 and ϕ_2 be spin-zero objects coupled to the divergences of J_1 and J_2. From the above discussion of nonconserved currents, one sees that we can write unsubtracted dispersion relarelations for $A^{\theta_2\theta_1}$, $A^{\phi_2\theta_1}$, $A^{\theta_2\phi_1}$, and $A^{\phi_2\phi_1}$. In analogy with (3) one can also, if he wishes, define a tensor amplitude $A^{\mu\nu}$, but it is clear from the above discussion that this amplitude will have a worse asymptotic behavior[8] than, say, $A^{\phi_2\phi_1}$.

III. ALGEBRAIC PROPERTIES OF SELF-CONSISTENT CURRENTS: HEURISTIC ARGUMENT

In this section we will give a physical argument that the self-consistent currents produced in a bootstrap theory are closed under commutation and can be used to form a Lie algebra. In the next section, we will give a more mathematical proof, based on S-matrix theory, of this assertion.

As we shall see, the physics which allows us to conclude that the self-consistent currents form an algebra comes from the behavior of the currents when q^ν, the energy and momentum carried off by a θ particle, tends to zero. We will first consider some aspects of the limit $q_s{}^\nu \rightarrow 0$ in amplitudes $J_{ba}{}^s$ for $a \rightarrow b+\theta_s$, where J^s is a self-consistent scalar current. These results, while not of central interest in this paper, will be helpful when we consider the limit $q_s{}^\nu \rightarrow 0$ in second-order amplitudes $A_{ba}{}^{\theta_s\theta}$. It is these second-order amplitudes which will directly be used in establishing the algebraic properties of the currents.

Consider the amplitudes $J_{ba}{}^s(0)$ for the emission of a massless scalar particle θ_s. In the limit where $q^\nu \rightarrow 0$ and θ_s carries off no energy or momentum, these amplitudes can be thought of as amplitudes for "emission" of a spurion. Evidently, if $J_{ba}{}^s$ is self-consistent and does not vanish in the limit $q^\nu \rightarrow 0$, we have the possibility of self-consistent "emission" of a spurion. Let us see now what this means. In our bootstrap theory, the strong interaction S-matrix S_{ba} is determined from self-consistency equations which can be symbolized by

$$S_{ba}=F_{ba}(S)\,, \tag{4}$$

where F_{ba} is a (highly nonlinear) functional of S. Suppose we have some solution $S_{ba}{}^0$ to (4). To check the stability of this solution, we try nearby solutions of the form $S_{ba}{}^0+\epsilon S_{ba}{}^1$ where $|\epsilon|\ll 1$. Neglecting terms of order ϵ^2, this leads to an equation

$$S_{ba}{}^1=\sum_{cd} A_{ba;dc}(S^0)S_{dc}{}^1\,, \tag{5}$$

$$A_{ba;dc}\equiv \delta/F_{ba}/\delta S_{dc}\,, \tag{6}$$

which must have no solution if the strong bootstrap is stable. Now, the point of this is that $S_{ba}{}^1$ can be thought of as the amplitude for emission of a spurion and Eq. (5) can be read as the condition that the spurion be self-consistent. It is clear then that if there is a self-consistent scalar current which gives rise, in the limit $q_\nu \rightarrow 0$, to a self-consistent spurion, there must be a solution to (5) and the strong interaction bootstrap will be unstable.[9]

We will now use similar reasoning to show that the self-consistent currents have certain algebraic properties.

Consider an arbitrary self-consistent current J which satisfies the linear bootstrap equation

$$J=X[J]\,. \tag{1}$$

Now we showed above that if the amplitudes $J_{ba}{}^s$ associated with a scalar current do not vanish as $q_s{}^\nu \rightarrow 0$, then strong-interaction Eq. (4) would be unstable. Let us investigate the analogous question of stability for Eq. (1). Evidently, we should consider second-order amplitudes for processes like $a \rightarrow b+\theta+\theta_s$, where θ couples to J and θ_s couples to a scalar current. By now it should be clear that as $q_s{}^\nu \rightarrow 0$ ($q_s{}^\nu$=four momentum of θ_s), the amplitude for $a \rightarrow b+\theta+\theta_s$ becomes equivalent to the amplitude for $a \rightarrow b+\theta$ +(spurion) and if the latter amplitude is nonzero, Eq. (1) is unstable.

Actually, we should be somewhat more precise here. We recall that, according to the discussion of Sec. II, the complete amplitude for $a \rightarrow b+\theta+\theta_s$ may contain some nonbootstrap terms. If it does, the resulting

---

[8] If a and b are single-particle states and $s=(p_a+q_1)^2=(p_b+q_2)^2$, then the tensor amplitude $A^{\nu\mu}$ will contain a term $(q_1{}^\nu q_2{}^\mu/q_1{}^2q_2{}^2)A^{\varphi_2\varphi_1}$ which goes as $(s/q_1{}^2q_2{}^2)A^{\varphi_2\varphi_1}$ for large s.

[9] Since we want our bootstrap to be stable, we require that $J_{ba}{}^s \rightarrow 0$ as $q^\nu \rightarrow 0$. A more complete derivation of this result, and some consequences which follow from it, will be discussed in the third paper of this series.

amplitude for $a \to b+\theta+$spurion will also contain some nonbootstrap terms and therefore does not directly imply instabilities in the bootstrap equation (1). To avoid this difficulty, we should work with the amplitude $A_{ba}{}^{\theta_s\theta}$ defined in Sec. II which contains only hadrons as internal particles and therefore contains no nonbootstrap terms. One can easily convince himself, then, that the correct statement is: if

$$\lim_{q_s^\nu \to 0} A_{ba}{}^{\theta_s\theta} \neq 0\,,$$

Eq. (1) is unstable.

Now if solution J of Eq. (1) is unstable, (1) has other nearby solutions of form $J+\epsilon J'$. Since Eq. (1) is linear and homogeneous in J, the formal requirement that $J+\epsilon J'$ satisfy (1) to order ϵ is simply

$$J' = X[J']. \tag{7}$$

In other words, J' must itself be a self-consistent current.

Returning to the amplitudes $A_{ba}{}^{\theta_s\theta}$ for $a \to b+\theta+\theta_s$, where θ couples to any self-consistent current J and θ_s couples to a self-consistent scalar current J_s, we arrive at the following conclusion. In the limit $q_s \to 0$, either these amplitudes vanish or else $J_{ba}' = \lim_{q_s^\nu} A_{ba}{}^{\theta_s\theta}$ is another self-consistent current.

Thus far we have shown how, given a self-consistent scalar current J^s and an arbitrary self-consistent current J, we can obtain a third self-consistent current J'. Now we will show how this relates to algebraic properties of the currents. We shall proceed in a formal way, leaving a discussion of the finer mathematical points to the next section.

Let us suppose that θ_s couples to the divergence $\partial_\nu J''^\nu$ of some nonconserved vector current J''^ν. A standard representation for $A_{ba}{}^{\theta_s\theta}$ is then

$$A_{ba}{}^{\theta_s\theta} = -i\int e^{iq_s\cdot x}\langle b|\,T[\partial_\nu J''^\nu(x)J(0)]\,|a\rangle d^4x\,,$$

where T is the time-ordering operator. Taking the limit $q_s^\nu \to 0$ we obtain

$$\lim_{q_s\to 0} A_{ba}{}^{\theta_s\theta} = -i\int \langle b|\,T[\partial_\nu J''^\nu(x)J(0)]\,|a\rangle d^4x\,, \tag{8a}$$

$$= -i\int \langle b|\,T[\partial_0 J''^0(x)J(0)]\,|a\rangle d^4x\,, \tag{8b}$$

$$= -i\langle b|\left[\int d^3x J''^0(\mathbf{x},0), J(0)\right]|a\rangle. \tag{8c}$$

Now, we have seen that $\lim_{q_s\to 0} A_{ba}{}^{\theta_s\theta}$ either vanishes or is equal to the matrix element $J_{ba}' = \langle b|J'(0)|a\rangle$ of another self-consistent current. We have thus discovered that in a bootstrap, if J''^ν is a self-consistent vector current and J is an arbitrary self-consistent current, then the equal-time commutator

$$[\textstyle\int d^3x' J''^0(\mathbf{x}',t)J(\mathbf{x},t)]$$

either vanishes or defines another self-consistent current J', according to

$$J'(\mathbf{x},t) = i\left[\int J''^0(\mathbf{x}',t)d^3x', J(\mathbf{x},t)\right]. \tag{9}$$

One generalization of this result is immediate. If If J''^ν is an axial-vector rather than a vector current, the proof that J' is a self-consistent current would proceed in exactly the same way.

Let us examine the content of these conclusions. We have seen how, given two self-consistent currents [i.e., J's satisfying (1)], a third can be obtained by evaluating a certain commutator. The set of all self-consistent currents is in this sense closed under commutation. More specifically, let $J_i{}^\alpha$, $i=1, 2, \cdots$ be the set of all independent vector and axial currents satisfying (1) and let

$$F_i(t) = \int J_i{}^0(\mathbf{x},t)d^3x; \tag{10}$$

then we must have

$$[F_i(t), J_j{}^\nu(\mathbf{x},t)] = i\sum_k C_{ijk}J_k{}^\nu(\mathbf{x},t) \tag{11}$$

and

$$[F_i(t), F_j(t)] = i\sum_k C_{ijk}F_k(t)\,, \tag{12}$$

where the C_{ijk}'s are constants. In short, the F's form a Lie algebra.

If we knew how to solve the strong interaction problem, we could calculate X and find out exactly what algebra is generated by the self-consistent currents. Obviously, this is out of the question at the present time.[9] But, noting that the observed weak and electromagnetic interactions of the hadrons should use some of these currents, one can look at experiment and try to make some guesses. As Gell-Mann and Ne'eman[10] have shown, the most economical group containing the observed weak and electromagnetic currents is $SU(3) \otimes SU(3)$. It is possible that this could be the complete algebra of the F's, or it might be just a subalgebra.[11]

IV. ALGEBRAIC PROPERTIES OF SELF-CONSISTENT CURRENTS: ARGUMENT FROM S-MATRIX THEORY

In this section we give a more mathematical proof that the self-consistent currents are, in the sense described in Sec. III, closed under commutation. Later

[10] M. Gell-Mann and Y. Ne'eman, Ann. Phys. (N.Y.) **30**, 360 (1964).

[11] Thus we have no immediate way of telling whether the Lie algebra will close on a small finite number of generators, a large finite number, or an infinite number. The hope is that there are at least subalgebras with a small finite number of generators. In a bootstrap theory this is a dynamical question relating to the number and character of self-consistent currents.

in the section we show how the restriction that the integrated object be the time component of a vector or axial object can be removed. In particular, we will see that if the integrated object is a space component of a current, the commutator will still yield another self-consistent current. This will allow us to enlarge the Lie algebra generated by the F operators.

To start the proof that we have a Lie algebra, we take an arbitrary self-consistent current J_1 and a self-consistent vector or axial current J_2^ν and define the matrix elements of an object K by

$$\langle b|K(0)|a\rangle \equiv i\langle b|\left[\int J_2^0(\mathbf{x},0)d^3x, J_1(0)\right]|a\rangle, \tag{13}$$

where the commutator is, of course, defined in terms of a sum over intermediate states. Here J_1, J_2^ν, and K correspond to J, J''^ν, and J' of Sec. III, and we shall consider quantities K_{ba} which as usual equal $\langle b|K(0)|a\rangle$ apart from kinematic factors. In the previous section, we gave a physical argument that K is another self-consistent current. We now want to show directly and, insofar as possible, rigorously that the quantities K_{ba} satisfy the unsubtracted dispersion relations which characterize a self-consistent current in S-matrix theory.

First let us note that if J_1 is self-consistent and J_2^ν is conserved, then the proof that K is self-consistent is trivial. For example, if J_2^ν is a component of the isospin current, $\int J_2^0 d^3x$ takes a particle only to another member of the same isospin multiplet. Then the commutator in the definition (13) of K reduces to an isospin rotation of J_1, and K must therefore be self-consistent if J_1 is. Thus, we can go on to the more interesting case where J_2^ν is not conserved.

Taking the physical arguments of Sec. III as a guide, let us study the amplitudes for $a+\theta \to b+\varphi$, where θ couples to J_1 and φ couples to $\partial_\alpha J_2^\alpha$ (φ corresponds to θ_s of Sec. III).[12] First, let us remind ourselves of the definition given in Sec. II of $A_{ba}^{\varphi\theta}$. We recall that $A_{ba}^{\varphi\theta}$ is that part of the amplitude for $a+\theta \to b+\varphi$ which contains only hadrons as internal particles. In a bootstrap theory $A_{ba}^{\varphi\theta}$ will be free of undetermined subtractions.

In Eqs. (8a)–(8c) of the previous section we indicated formally that as the four-momentum q^ν of φ tends to zero the amplitude $A_{ba}^{\varphi\theta}$ reproduces the ba matrix of the commutator; specifically

$$K_{ba} = \lim_{q^\nu \to 0} A_{ba}^{\varphi\theta}. \tag{14}$$

Our first step in proving that K is a consistent current is to show this more rigorously. In particular, we must establish that the formal relation (8a) for the amplitude $a+\theta \to b+\varphi$ does not differ from $A_{ba}^{\varphi\theta}$ by some non-bootstrap subtraction terms. For the case where a and b are single particle states, we will show that (14) does, in fact, hold. It will then be assumed that (14) holds in general.

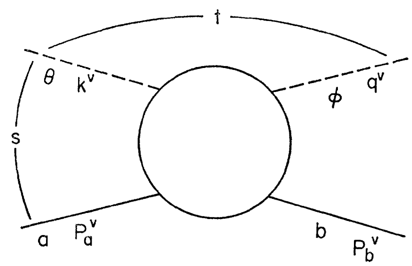

FIG. 2. Kinematics for the reaction $a+\theta \to b+\varphi$.

To this end, we study the reaction $a+\theta \to b+\varphi$, where a and b are single-particle states, using the kinematics shown in Fig. 2. The "mass" of θ is then $\sqrt{(k^2)}$ and, in view of the fact that we want the limit $q^\nu \to 0$, we set the "mass" $\sqrt{(q^2)}$ of φ equal to zero. We now write a fixed t dispersion relation for $A_{ba}^{\varphi\phi}$

$$A_{ba}^{\varphi\theta}(s,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Im}A_{ba}^{\varphi\theta}(s',t)ds'}{s'-s} \tag{15}$$

which, since we have by definition of $A_{ba}^{\varphi\theta}$ only hadrons (i.e., Regge objects) as internal particles, should converge for values of t sufficiently less than zero; for other values of t the integral is defined by analytic continuation. When $q^\nu \to 0$, we have $s \to M_b^2$, $t \to k^2$ and

$$\lim_{q^\nu \to 0} A_{ba}^{\varphi\theta}(s,t) = A_{ba}^{\varphi\theta}(M_b^2,k^2) = \frac{1}{\pi}\int \frac{\operatorname{Im}A_{ba}^{\varphi\theta}(s',k^2)ds'}{s'-M_b^2}. \tag{16}$$

Using a technique due to Fubini and Furlan,[13] one can now show that the dispersion integral on the right of (16) is equal to the b-a matrix element of the commutator (13). Here we will give only a sketch of the proof; the details of the method are amply illustrated in Ref. 13.

Let us assume, for simplicity, that J_1 is a scalar current. Taking covariant normalizations for the states $\langle a|$ and $\langle b|$, i.e., $\langle \mathbf{p}|\mathbf{p}'\rangle = 2p^0\delta(\mathbf{p}-\mathbf{p}')$, we then have, on invariance grounds, $\langle b|K(0)|a\rangle = K_{ba}(k^2)$, where K_{ba} is a scalar function if $k^2 = (p_a - p_b)^2$. Evaluating (13) by inserting a complete set of intermediate states then gives

$$K_{ba} = i\sum_n \left[\langle b|\int J_2^0(\mathbf{x},0)d^3x|n\rangle\langle n|J_1(0)|a\rangle - \langle b|J_1(0)|n\rangle\langle n|\int J_2^0(\mathbf{x},0)|a\rangle\right].$$

[12] We switch from the notation θ_s to φ in order to be in accord with the notation of Sec. II and that which will be used in Sec. V, where φ couples to the divergence of a current.

[13] S. Fubini and G. Furlan, Physics 4, 229 (1965).

Next, using

$$\langle b|\int J_2{}^0(\mathbf{x},0)d^3x|n\rangle$$
$$=i^{-1}(p_b{}^0-p_n{}^0)^{-1}\langle b|\int \partial_\nu J_2{}^\nu(\mathbf{x},0)d^3x|n\rangle$$

and performing some algebraic manipulations yields

$$\langle b|\int J_2{}^0(\mathbf{x},0)d^3x|n\rangle\langle n|J_1(0)|a\rangle$$
$$=-\frac{1}{\pi}\frac{1}{s'-M_b}\frac{(p_n{}^0+p_b{}^0)}{2p_n{}^0}\mathrm{Im}A_{ba}{}^{\varphi\theta}(s',t,q^2)\,,$$

where $\sqrt{(s')}$ is the mass of the state $|n\rangle$ and $A_{pa}{}^{\varphi\theta}$ is the invariant amplitude defined above except that here φ has a mass of $q^2=(p_n{}^0-p_b{}^2)$. Suppressing, for the moment, the second term in the commutator we find

$$K_{ba}(k^2)=-\frac{1}{\pi}\int_0^\infty \mathrm{Im}\frac{A_{ba}{}^{\varphi\theta}(s',k^2,q_s{}^2)}{s'-M_b{}^2}\left\{\frac{p_n{}^0+p_b{}^0}{2p_n{}^0}\right\}ds'$$
$$-(\text{second term}).\quad(17)$$

Now, following Fubini and Furlan, we note that the left-hand side is a Lorentz scalar, so that we are free to choose the frame in which we evaluate the integral. Everything inside the integral is an invariant except q^2, which is $(p_n{}^0-p_b{}^0)^2$, and the factor in curly brackets. One can verify that if we choose a frame such that $\mathbf{p}_a\to\infty$ and $\mathbf{p}_b\to\infty$, then $q^2\to 0$ and the factor in brackets tends to unity. If this limit can be taken inside the integral on the right-hand side of (17), we will clearly obtain the part of the integral in (16) running over the positive s' axis. The interchange of limit and integral sign will be justified if (16) coverges. For k^2 sufficiently less than zero, (16) should converge, and other values of k^2 can then be reached by analytic continuation. A similar calculation shows that the second term in the commutator produces the integral over the negative s' axis in (16), and we conclude that $K_{ba}(k^2)=A_{ba}{}^{\varphi\theta}(M_b{}^2,k^2)$ which is the desired result.

We have now explicitly verified (14) for the case that a and b are single-particle states. Taking this result together with the formal relations (8a)–(8c), it is reasonable to assume that (14) holds for arbitrary states a and b, and we shall do so.

Since $A_{ba}{}^{\varphi\theta}$ is defined so that it satisfies unsubtracted dispersion relations, we learn from (14) that K_{ab} also satisfies unsubtracted dispersion relations. It remains then to show that the discontinuities which appear in the dispersion relations for K are those which characterize a current in S-matrix theory. Now, in our bootstrap approach to the currents we are implicitly assuming that all the discontinuities in our scattering amplitudes are determined by unitarity in the various channels. Thus if we can show that K has the unitarity properties characteristic of a current, we will know that the unsubtracted dispersion relations satisfied by K are those which define a self-consistent current.

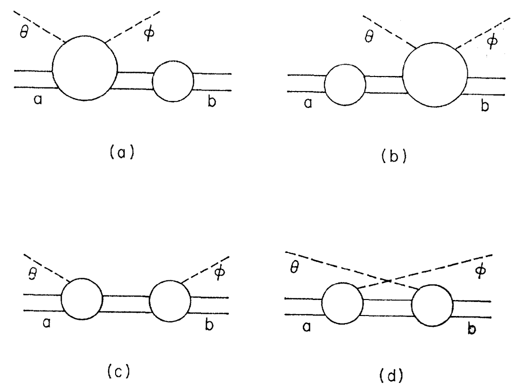

FIG. 3. Some typical terms in the unitarity relation for the process $a+\theta\to b+\varphi$. The states a and b represent arbitrary systems of hadrons. All solid lines are hadrons.

It is not hard to verify that K_{ba} has the correct unitarity properties. We begin with the unitarity condition for $A_{ba}{}^{\varphi\theta}$ and then let $q^\nu\to 0$. Figure 3 shows some typical terms in the unitarity condition for $A_{ba}{}^{\varphi\theta}$. In the limit of vanishing q^ν the amplitudes where φ and θ come off the same blob become, according to (14), equal to matrix elements of K. Thus the terms in Figs. 3a and 3b become, in this limit, the expected linear unitarity terms for a current. Evidently, the diagrams in Figs. 3c and 3d must vanish. To see that they do, one need only recall that according to Sec. III the stability of the bootstrap requires all the amplitudes for (hadrons) $\to$ (hadrons) $+\varphi$ to vanish as $q_s{}^\nu\to 0$. Thus we conclude that K_{ba} has the desired unitarity properties.

As pointed out above, it then follows that the object K defined by (13) is, in fact, a self-consistent current.

So far, we have restricted ourselves to commutators like (9) where the integrated object is the time component of a vector or axial current. This restriction is easily removed. To see how, we first note that for a current $J_1{}^{\mu_1\mu_2\cdots\mu_n}$ with n Lorentz indices, one could prove that the object $K^{\mu_2\cdots\mu_n}$ defined by

$$K^{\mu_2\cdots\mu_n}(\mathbf{x},t)\equiv\left[\int J_1{}^{0,\mu_2\cdots\mu_n}(\mathbf{x},t)d^3x,J_2(\mathbf{x},t)\right]\qquad(18)$$

is a self-consistent current. (We suppress any Lorentz indices carried by J_2.) The proof would proceed essentially in the same way as before except that the particle φ which couples to $\partial_\nu J^{\nu,\mu_2\cdots\mu_n}$ will now have spin. The only point in the proof where the spin of φ could cause difficulty is in our argument that K satisfies the correct unitarity conditions. There we used stability of the strong interactions to argue that all the amplitudes for $a\to b+\varphi$ vanish as $q^\nu\to 0$. This need not be true if φ has spin, but in any case φ couples to the divergence of a current so that the amplitudes for $a \rightarrow b+\varphi$ must vanish as $q^\nu \rightarrow 0$ except possibly for infrared terms where φ couples to an external line. In cases where these infrared terms appear, direct analysis of the unitarity conditions for K is complicated, but in perturbation theory the unitarity conditions still hold and there is no reason to believe that the infrared divergent terms will cause any difficulty. It follows then, that our Lie algebra based on the space integrals of the time components of vector and axial vector currents can be enlarged to include all operators of the form[14]

$$F^{\mu_2\cdots\mu_n}(t)=\int J^{0,\mu_2\cdots\mu_n}(\mathbf{x},t)d^3x\,, \tag{19}$$

where $J^{\mu_1,\mu_2\cdots\mu_n}$ is a self-consistent current. A particular consequence of this comes from the fact that if J_α is self-consistent, so is $\epsilon^{\alpha\beta\gamma\delta}J_\delta$. We can then form

$$F^{\beta\gamma}(t)=\int \epsilon^{0\beta\gamma\delta}J_\delta(\mathbf{x},t)d^3x\,, \tag{20}$$

which brings the integrals of the space components of vector and axial currents into the algebra.

The relationship of these algebraic properties of the self-consistent currents to symmetries will be discussed in the following sections. Before proceeding, however, we would like to point out two important matters of principle relating to current algebras.

(i) The linear equations for the J's determine the self-consistent currents only up to a scale factor. As has been pointed out by Gell-Mann,[1] the nonlinear commutation relations can be used to specify these scale factors.

(ii) No one really knows to what extent the bootstrap can determine the strong interactions. It may, in in fact, have many solutions. To every solution there will be a set of self-consistent currents, and according to the conclusions of this section, to every solution there will correspond some Lie algebra generated by these currents. If there are many solutions to the strong-interaction bootstrap an especially clean way to pick a particular solution would be to specify that the associated currents have certain algebraic properties. We refrain from speculating on how much, if any, information about the current algebra would have to be fed in.

V. CONVERGENCE OF THE COMMUTATORS

This section is devoted to a discussion of the convergence properties of the equal-time commutators which appear in current algebras. By the convergence of a commutator, we mean, of course, the rapidity with which the sum over intermediate states involved in the definition of a commutator converges. We will discuss only the commutators of pairs of the F operators defined in the last section. Further, we will restrict ourselves to F's which are integrals of the various space and time components of vector or axial currents. Commutators involving more complicated F's or an F and a current density can be treated in a similar fashion.

We employ the notation

$$F_i^\nu=\int J_i^\nu d^3x\,,\quad \nu=0,1,2,3\,, \tag{21}$$

where i is an isotopic index.

Consider the b-a matrix element of the commutator

$$\langle b|[F_i^\nu,F_j^\mu]|a\rangle=\sum_n [b|F_i^\nu|n\rangle\langle nF_j^\mu|a\rangle -\langle b|F_j^\mu|n\rangle\langle n|F_i^\nu|a\rangle. \tag{22}$$

Since the F's are space integrals of current components, we can write

$$\langle b|F_i^\nu|a\rangle=\delta^3(\mathbf{p}_a-\mathbf{p}_b)f_{ba}{}^{i\nu}(\mathbf{p})\,, \tag{23}$$

where $\mathbf{p}=\mathbf{p}_a=\mathbf{p}_b$. The right-hand side of (22) can then be rewritten as

$$\sum_n[f_{bn}{}^{i\nu}(\mathbf{p})f_{na}{}^{j\mu}(\mathbf{p})-f_{bn}{}^{j\mu}(\mathbf{p})f_{na}{}^{i\nu}(\mathbf{p})]\,, \tag{24}$$

which should hold for any $\mathbf{p}$; we note that, in general, the f's will depend on $\mathbf{p}$. We will discuss the convergence of (24) for two cases: first $\mathbf{p}=0$ and then the limit $\mathbf{p}\rightarrow\infty$. The advantages of using the later case have been stressed by Fubini and Furlan.[13]

The key to the discussion will be the convergence properties of matrix elements of nonconserved currents which were outlined in Sec. II. We recall that a nonconserved current was defined as the sum of spin-one and spin-zero pieces J_1^ν and J_0,

$$J_{ba}{}^\nu=J_{1,ba}{}^\nu-i(q^\nu J_{0,ba}/q^2)\,. \tag{3}$$

In our bootstrap approach J_1^ν and J_0 are objects which satisfy unsubtracted dispersion relations and have the asymptotic behavior of strong interaction amplitudes. Because of the q^ν multiplying J_0 in Eq. (3), J^ν will, for fixed q^2, have *worse* asymptotic behavior. Equation (3) can, of course, be inverted to obtain, apart from kinematic factors,

$$J_{0,ba}=\langle b|\partial_\alpha J^\alpha|a\rangle\,,$$

and

$$J_{1,ba}{}^\nu=\langle b|J^\nu-\frac{\partial^\nu}{\Box^2}\partial_\alpha J^\alpha|a\rangle\,,$$

To begin our discussion of (24) for $\mathbf{p}=0$, let us see what, in this case, the f's correspond to in terms of scattering amplitudes. For $\nu=0$ we have

$$\langle n|F_i^0|a\rangle=\langle n|\int J_i^0d^3x|a\rangle =i(M_n-M_a)^{-1}\langle n|\int \partial_\nu J_i^\nu d^3x|a\rangle \tag{25}$$

[14] One should note that unless $\partial_\nu J^{\nu\,\mu_2\cdots\mu_n}=0$, $F^{\mu_2\cdots\mu_n}$ is *not* a covariant quantity with simple Lorentz transformation properties.

and f_{na}^{i0} is, apart from kinematic factors, an energy denominator times the amplitude for $a \rightarrow n+\varphi$, where φ_i is a particle which couples to $\partial_\nu J_i^\nu$. We note that the mass-squared of φ_i is given by $q^2=(M_n-M_a)^2$. (For the present case of $\mathbf{p}=0$, q^2 does not remain fixed but varies with the intermediate state.) On the other hand, if ν is a space component, we have

$$\langle n|F_i^k|a\rangle=\langle n|\int J_i^k d^3x|a\rangle \tag{26}$$

and f_{na}^{ki} is simply kinematic factors times the amplitude for $a \rightarrow n+\theta_i$, where θ is a particle coupled to the transverse part of J. Note that for $\nu=1, 2, 3$ we do *not* pick up the energy denominator of Eq. (25). The mass-squared of θ is again, of course, $q^2=(M_n-M_a)^2$.

In our bootstrap theory, where all form factors are supposed to vanish as $q^2 \rightarrow \infty$, it is not hard to see that with $\mathbf{p}=0$, the commutator (24) should converge quite rapidly. We need only recall here that the $f_{na}^{i\nu}$ correspond to the emission of a θ or φ particle with $q^2=(M_a-M_n)^2$. Then if all form factors go more or less like q^{-2} for large q^2, the product of form factors entering into (24) will go something like M_n^{-4} for high mass states n and it is clear that (24) will converge rapidly. For $\nu=0$, we gain additional convergence from the energy denominator $(M_a-M_n)^{-1}$ in (25). We note in passing that in a nonbootstrap theory where form factors need not vanish as $q^2 \rightarrow \infty$, the prospects for rapid convergence of (24) are considerably less favorable.

Since for $\mathbf{p}=0$, the commutator (24) can be expected to converge rapidly, it would be tempting to assume that the sum over intermediate states is dominated by the contributions of a few single-particle and resonant states. The conditions under which this approximation will be valid are, however, more involved than one might at first expect. To see why, let us consider a two-particle intermediate state $|cd,\mathbf{k}\rangle$, where $\mathbf{k}$ is the center-of-mass momentum of cd. Now if there is a resonance at $|\mathbf{k}|=k_r$ in the cd system, the amplitudes $f_{(cd)a}^{i\nu}$ and $f_{b(cd)}^{i\nu}$ will be very large for $|k|$ near k_r and the resonance will, of course, make a large contribution to (24). However, this is not the only way that amplitudes like $f_{(cd)a}^{i\nu}$ can become large. Recalling that the $f_{(cd)a}^{i\nu}$ corresponds to $a+\theta(\varphi) \rightarrow c+d$ with $q^2=(M_a-M_{cd})^2$ ($M_{cd}=$total c.m. energy of c and d), we note that for energies M_{cd} such that $q^2=(M_{cd}-M_a)^2$ is near a resonance in the θ_i or φ_i form factors, $f_{(cd)a}^{i\nu}$ can again become quite large even though there is no resonance in the cd system. The fact that these peaks in form factors might make an important contribution to equations like (24) was first noted by Bietti,[15] who has explicitly shown that their contribution can be as large as the contribution of resonant intermediate states. Since q^2 is positive and growing as the mass of the intermediate state increases, it appears *inescapable*, in fact, that peaks or lumps in the form factors will be encountered before rapid convergence sets in. Clearly, in cases where peaks in form factors make a large contribution to the commutator, it would be incorrect to suppose that single particle intermediate states dominate. For commutators involving an F^{i0}, however, there will be situations where these states do dominate. This will come about if any resonances in the form factors appear at a large enough mass so that the energy denominator $(M_a-M_n)^{-1}$ of (25) will damp their contribution relative to the contribution of single-particle and resonant states with M_n near M_a.

Fubini and Furlan[13] have suggested that it is often best to evaluate (24) in the limit $\mathbf{p} \rightarrow \infty$. We will now investigate the convergence of the commutator (24) in this limit, restricting ourselves to the case where a and b are single particle states. We will find that if we take $\mathbf{p} \rightarrow \infty$ along the "3" axis, commutators involving F^{i0} and F^{i3} converge considerably more rapidly than commutators involving F^{i1} or F^{i2}. For $\mathbf{p}=\infty$ the form factor bumps do not appear, and for commutators which converge rapidly, single-particle and resonance states will dominate. The implications of this result for $SU(6)$ will be discussed in the final section.

First we consider the commutator of two F^{i0}'s. In essentially the same way as we verified Eq. (13) in the last section, one finds that in the limit $\mathbf{p} \rightarrow \infty$ the right-hand side of (24) becomes[13]

$$\int_{-\infty}^{\infty} \frac{\operatorname{Im} A_b{}^{\varphi_j}{}_a{}^{\varphi_i}(s',0)}{(s'-M_a{}^2)(s'-M_b{}^2)} ds', \tag{27}$$

where φ_i is a massless particle coupling to $\partial_\nu J_i^\nu$ and $A_b{}^{\varphi_j}{}_a{}^{\varphi_i}(s,t)$ is the amplitude, as defined in Sec. II, for $\varphi_i+a \rightarrow \varphi_j+b$. Now according to the discussion of Sec. II, $A^{\varphi_j\varphi_i}$ will have the same asymptotic behavior as a strong interaction amplitude. Thus the integral (27) should behave like a once-subtracted forward dispersion relation for a strong interaction amplitude and will therefore converge well.

It is perhaps worth recalling here our previous remark that in a nonbootstrap theory, there is little reason to believe that $A_b{}^{\varphi_j}{}_a{}^{\varphi_i}$ will have an asymptotic behavior like that of a strong interaction amplitude. In the first place, since we are treating the couplings of φ_i and φ_j to first order, there is no unitarity bound on $A_b{}^{\varphi_j}{}_a{}^{\varphi_i}$, and secondly, one should remember that φ_1 and φ_2 couple to the divergences of currents. In a usual sort of field theory these "gradient couplings" would be likely to endow $A_b{}^{\varphi_j}{}_a{}^{\varphi_i}$ with a singular asymptotic behavior. Again, we see that the prospects for rapid convergence of the commutators are more favorable in a bootstrap theory.

We now consider commutators, still in the limit $\mathbf{p} \rightarrow \infty$, involving F_i^k, $k=1, 2, 3$. Using the identities

$$F_i^k = \tfrac{1}{2} \sum_{l,m=1}^{3} \epsilon^{klm} \int \epsilon^{0lm\nu} J^\nu d^3x \tag{28}$$

[15] A. Bietti, Phys. Rev. 144, 1289 (1966).

and

$$\langle \mathrm{n}|\int \epsilon^{0lm\nu}J^{\nu}|a\rangle = i(p_n{}^0 - p_a{}^0)^{-1}\langle n|\int \epsilon^{\mu lm\nu}\partial_\mu J_\nu|a\rangle$$

the b-a matrix elements of the commntator $[F_i{}^k, F_j{}^{k'}]$ can be, by taking the limit $\mathbf{p}\to\infty$, converted into an integral of the form

$$\int_{-\infty}^{\infty} \frac{\mathrm{Im}A_b{}^{\varphi_i{}^k}{}_a{}^{\varphi_j{}^{k'}}(s',0)}{(s'-M_a{}^2)(s'-M_b{}^2)}ds', \tag{29}$$

where, for example, the massless particle $\varphi_i{}^k$ now couples to $\frac{1}{2}\epsilon^{klm}\epsilon^{\nu lm\mu}\partial_\nu J_\mu$. The integrand of (29) contains the "squares" of terms like

$$\tfrac{1}{2}\epsilon^{klm}\epsilon^{\nu lm\mu}q_\nu\langle n|J_{\mu i}|a\rangle. \tag{30}$$

Let us investigate the asymptotic behavior of these amplitudes. We assume that the limit $\mathbf{p}\to\infty$ has been taken with $\mathbf{p}$ directed along the 3 axis. Then, since $q\cdot q=0$, we have $q^1=q^2=0$ and $q^0=q^3$. For $k=3$, (30) is then

$$\begin{aligned} q_0\langle n|J_{3i}|a\rangle - q_3\langle n|J_{0i}|a\rangle &= q_0\langle n|J_{0i}|a\rangle - q_3\langle n|J_{3i}|a\rangle,\\ &= q^\nu\langle n|J_{\nu i}|a\rangle, \end{aligned} \tag{31}$$

where we have used $q^0=q_0=q\nu=-q_3$. Thus the particle $\varphi_i{}^3$ actually behaves as if it were coupled to $\partial_\nu J_i{}^\nu$, and the amplitudes involving $\varphi_i{}^3$ will converge as described above. The situation is different, however, for $k=1$ or 2. In these two cases (30) becomes

$$-q_0\langle n|J_{2i}|a\rangle \tag{32}$$

and

$$q_0\langle n|J_{1i}|a\rangle, \tag{33}$$

respectively. Now since q_1 and q_2 are zero, $\langle n|J_{2i}|a\rangle$ and $\langle n|J_{1i}|a\rangle$ are equivalent to matrix elements of the spin-one part of the current $J_{1i}{}^\nu = J_i{}^\nu - (\partial^\nu\partial_\mu/\Box^2)J^\mu$ of Eq. (3). Furthermore, we know that amplitudes involving the spin-one part of a current will behave asymptotically like strong interaction amplitudes. Thus the objects in (32) and (33) will go like q^0x (strong interaction amplitude). For large s, $q^0\sim\sqrt{s}$ and it is evident that dispersion integrals (29) for commutators involving $k=1$ or 2 will converge less rapidly.

Let us summarize these results, keeping in mind that we have taken $\mathbf{p}\to\infty$ along the 3 axis. We have found that the dispersion integrals for commutators involving only F^{i3}'s and F^{i0}'s should converge rapidly. If the commutator involves one F^{i1} or F^{i2}, the integral will, for large s, contain an extra factor of $\sqrt{s}$, and if two of these latter objects are involved, the integrand will be multiplied by $\sqrt{(s)}^2=s$.

In discussing the commutators for $\mathbf{p}=0$, we pointed out that although (24) should converge rapidly, single-particle and resonant intermediate states were not likely to dominate because of form-factor difficulties, except in some cases involving F^{i0}'s. This situation does not occur for $\mathbf{p}\to\infty$ where all our φ particles have a fixed mass of zero. If we are dealing with $\int J^0d^3x$'s or $\int J^3d^3x$'s so that the dispersion relations converge rapidly, it would seem most reasonable to assume that the single particle and resonant states dominate. We do, however, pay a price for this. For $\mathbf{p}=0$, the matrix element for $\langle n|\int J^0d^3x|a\rangle$ vanishes unless n has the same spin as a. This is clearly not the case for $\mathbf{p}\to\infty$, where there can be orbital momentum between a and φ.

VI. CONNECTION WITH SYMMETRIES

We have seen how the bootstrap could produce a set of self-consistent currents which can be used to form a Lie algebra. Since this algebra would be an intrinsic property of the strong interaction bootstrap, it is natural to ask if this algebra might reflect some approximate symmetry of the strong bootstrap. In this section, we investigate the conditions under which this will be the case. We will find that the prospects for a correlation between current algebras and symmetry are quite favorable in the bootstrap theory. Later in the section we will argue that a close correlation between current algebras and approximate symmetry would seem rather unlikely in a theory with essential nonbootstrap elements.

It is perhaps worth remarking here that our point of view on the connection between current algebras and symmetries would be that both are produced by the bootstrap and neither need be thought of as the "cause" of the other.

Let us then consider a closed algebra of F operators satisfying

$$[F^i, F^j] = i\textstyle\sum_k C^{ijk}F^k. \tag{12}$$

This could be the complete algebra of the F's or just a subalgebra. Our first task will be to study the conditions under which one can expect the hadrons to fall into supermultiplets associated with the algebra (12). We will then argue that these conditions are likely to be satisfied in a bootstrap. By itself the existence of supermultiplets does not, of course, constitute a symmetry; we must also know that the couplings among the particles are symmetric, and this will be studied later in the section.

First let us define exactly what we mean by a supermultiplet associated with the algebra of Eq. (12). We define a supermultiplet as a set of physical, single-particle states $|\alpha\rangle$, $\alpha=1\cdots N$, which have the properties (we do not distinguish between stable and unstable particles in this section):

(i) For some fixed $\mathbf{p}$, such as $\mathbf{p}=0$ or ∞, we have

$$\sum_{\alpha''=1}^{N}[f_{\alpha\alpha''}{}^i(\mathbf{p})f_{\alpha''\alpha'}{}^j(\mathbf{p}) - f_{\alpha\alpha''}{}^j(\mathbf{p})f_{\alpha''\alpha'}{}^i(\mathbf{p})] \approx i\sum_k C^{ijk}f_{\alpha\alpha'}{}^k(\mathbf{p}), \tag{34}$$

where $(2\pi)^3\delta(\mathbf{p}_\alpha-\mathbf{p}_{\alpha'})f_{\alpha\alpha'}{}^i(\mathbf{p})=\langle\alpha|F^i|\alpha'\rangle$, $\mathbf{p}=\mathbf{p}_\alpha=\mathbf{p}_{\alpha'}$ as in the preceding section.

(ii) The masses M_α, $\alpha=1\cdots N$, of the states are all in the same general neighborhood. More specifically, we require $|M_\alpha-M_{\alpha'}|\lesssim M_c$, where M_c is a mass characteristic of the strong interactions ($M_c\approx 1$ BeV).

(iii) The approximate representation $f_{\alpha\alpha'}{}^i$ of the algebra is irreducible.

In connection with (i), we would like to stress that (34) is not required to hold for all $\mathbf{p}$. For example, it is possible that some set of states $|\alpha\rangle$, $\alpha=1\cdots N$ will form a supermultiplet for $\mathbf{p}=\infty$ but not for $\mathbf{p}=0$ or vice versa; the reason being that, as we saw in the last section, the convergence properties of the commutators may change as we go from $\mathbf{p}=0$ to $\mathbf{p}=\infty$.

Although (iii) is customary in the definition of a supermultiplet, it is sometimes convenient to relax (iii) to

(iii′) The approximate representation $f_{\alpha\alpha'}{}^i(\mathbf{p})$ contains a small number of irreducible components.

As an example of the content of (iii) relative to (iii′) we note that in $SU(3)$, on account of ϕ-ω mixing, the physical vector mesons can satisfy (iii′) but not (iii).

Having defined our notion of a supermultiplet, let us see under what physical conditions supermultiplets can be expected to appear. There are two obvious requirements on the F^i's. If (i) is to hold, it is clear that:

(A) For the $\mathbf{p}$ under consideration, the commutator (34) must be dominated by single-particle intermediate states.

Given (i), a necessary and sufficient condition under which (ii) will hold is:

(B) $|f_{\alpha\alpha'}{}^i(\mathbf{p})|\ll 1$ if $|M_\alpha-M_{\alpha'}|\gtrsim M_c$.

We note that (B) is equivalent to the assumption of rapid convergence of the commutators.

Given (A) and (B), one further condition is necessary to get (iii′). To see its content, consider the following chain of thought. We take all single-particle states and label them according to $|x\rangle$, $x=1, 2, \cdots$. Assuming (A), we can write

$$\sum_{x''}(f_{xx''}{}^i f_{x''x'}{}^j - f_{xx''}{}^j f_{x''x'}{}^i)\approx i\sum_k C^{ijk} f_{xx'}{}^k \qquad (35)$$

so that the matrices $f_{xx'}{}^i$ form an approximate representation of our algebra. Let us reduce this representation. In doing so, we go from the particle basis $|x\rangle$ to a set of base states which are, in general, linear combinations of the true particle states. If we want our supermultiplets to always satisfy (iii) we would clearly have to require that the new base states be approximately the same as the old particle states. Actually, there is no reason why we should not admit some "slightly mixed" supermultiplets which satisfy (iii′) instead of (iii). But, even if we allow for some mixing by going to (iii′), it is clear that we will not obtain recognizable supermultiplets if in the course of reducing (35) we have to introduce too many states which are strong mixtures of the physical-particle states. Let us see what property of the hadrons could keep this from occurring. If (B) holds, we see that two states cannot mix if their masses are sufficiently different. Thus we want candidates for mixing to be generally spaced at intervals of M_c or greater. On the other hand, if all hadron states were spaced this far apart, we would have $f_{nn'}{}^i$ always ≈ 0 and there could be no supermultiplets. The key point is, now, that candidates for mixing will generally have the same quantum numbers, but two states belonging to the same irreducible representation will usually have different quantum numbers. We are thus led to require, in addition to (A) and (B), that the mass spectrum of the hadrons has the following property:

(C) *On the average*, the spacing between particles with the same quantum numbers is of order M_c or greater, while the spacing between states with different quantum numbers can be considerably less.

One can convince himself that (A), (B), and (C), are, for practical purposes, necessary and sufficient for the appearance of supermultiplets associated with a current algebra. When they hold we expect to find supermultiplets; in general, supermultiplets of the slightly mixed variety (iii′) but in especially favorable cases pure supermultiplets (iii).

Next we would like to argue that in a bootstrap, (A), (B), and (C) will often hold. In the previous section we showed that in certain (not uncommon) cases, the commutator of two F's will satisfy (A). The particular choices of F's and values of $\mathbf{p}$ for which this will be the case were discussed there and will not be repeated. As far as (B) is concerned, we have previously noted that it is equivalent to rapid convergence of commutators. Putting numbers into the formulas and verbal arguments of Sec. V, one can see that in our bootstrap theory, (B) could be expected to hold for M_c on the order of a nucleon mass. Finally, (C) should be a rule-of-thumb, qualitative characteristic of the mass spectrum in a bootstrap world, at least for the lower lying states. To see this, we recall that in a bootstrap, all particles are composite objects bound by forces associated with the exchange of other particles. We note that the range of most forces will typically be of order M_c^{-1}. Now suppose that for each set of quantum numbers only one channel were available, say πN for $Y=1$, $I=\frac{1}{2}$ or $\frac{3}{2}$, or $\pi\Xi$ for $Y=-1$, $I=\frac{1}{2}$ and $\frac{3}{2}$. If this were the case, it would not be hard to convince oneself that particles with the same quantum numbers would be spaced at intervals of about M_c. The reason is that a force with range M_c^{-1} cannot, unless it has some very complicated structure, produce two bound or resonant states in the same channel which have masses closer together than about M_c. In reality, of course, there are many channels for each set of quantum numbers, but for the low-lying states, only one or a few will generally be important, with the result that low-mass states with the same quantum numbers will usually be spaced by about M_c. There is, of course, no reason why states with different quantum numbers need be so far apart. For the higher mass states the situation is not so clear cut, but in a qualitative way one might expect (C) to hold rather generally in the bootstrap. We conclude then that the bootstrap theory presents a dynamical framework which is quite favorable for the appearance of supermultiplets associated with current algebras.

The appearance of supermultiplets will allow us to do two things. First we can classify the states according to irreducible (iii) or nearly irreducible (iii′) representations of the algebra. Secondly, we can easily find explicit expressions for the matrices $f_{\alpha\alpha'}{}^i(\mathbf{p})$. For an irreducible representation the latter are just Clebsch-Gordan coefficients. If the representation is mixed we need only add a few parameters to describe the mixing. The value in knowing the $f_{\alpha\alpha'}{}^i$'s is, of course, that they are matrix elements of currents which will often enter into physical amplitudes. Also, the Goldberger-Treiman relation[16] and its generalizations may often allow us to relate couplings such as α-α' meson to $f_{\alpha\alpha'}$'s.

We conclude our discussion of supermultiplets with a few quite obvious, but important remarks.

(i) Clearly, we can talk about supermultiplets associated with the algebra of Eq. (12) without having to imagine any dynamical limit in which the algebra corresponds to an exact symmetry of strong interactions.

(ii) For F's which are integrals of, for example, the z component of vector or axial currents, we saw, in Sec. V, that (A) is most likely to hold if one takes $\mathbf{p}\rightarrow\infty$ along the z axis. Thus algebras involving integrals of space components of currents might only lead to supermultiplets in the limit $\mathbf{p}\rightarrow\infty$.

(iii) We have been implicitly assuming that our algebra has finite-dimensional representations. This will not be the case if the algebra is noncompact. However, with only minor modifications our above treatment can be extended to infinite supermultiplets associated with noncompact algebras. The essential change would be that property (ii) in our definition of a supermultiplet would become $|M_\alpha - M_{\alpha'}| \lesssim M_c$ when α and α' are "neighboring states" in the infinite-dimensional representation $f_{\alpha\alpha'}{}^i$ of the algebra. The reader is referred to the papers of Dothan, Gell-Mann, and Ne'eman[17] and Dothan and Ne'eman[18] for a discussion of the properties and uses of non-compact algebras.

[16] The relationship of the Goldberger-Treiman relation and its generalization to our self-consistent current formalism is discussed in (I).

[17] Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965).

[18] Y. Dothan and Y. Ne'eman (to be published).

As pointed out above, the existence of supermultiplets does not, by itself, constitute a complete symmetry. Given the supermultiplets we have to know also that the interactions are symmetric. We now turn to this latter topic. In the discussion it will be convenient to divide current algebras into two classes.

(α) Algebras for which one can *imagine* a sensible dynamical limit in which the algebra is an exact symmetry of the strong interaction. By a sensible limit we mean, of course, one which does not violate any fundamental property such as Lorentz invariance or unitarity, which we expect the strong interaction to possess.

(β) Algebras for which one cannot imagine such a limit. As examples, $SU(3)$ would belong to (α) whereas $SU(6)$ seems to come under the heading of (β). We would like to stress the word "imagine" in (α). We do not mean to imply that the symmetry limit can be physically obtained by turning off some kind of symmetry-breaking interaction. Remember that we are working in a bootstrap theory and, just because the bootstrap has a solution which possesses a certain approximate symmetry, there is no reason to believe that it will have, in addition, a corresponding completely symmetric solution.

We begin with case (α). It will be assumed that we have an algebra which, for some fixed $\mathbf{p}$, splits the single-particle states into supermulitplets. We will further assume that the mass differences between the members of a supermultiplet are small and, to a zeroth approximation, can be neglected. From this assumption of degenerate supermultiplets we will find, in some cases, that self-consistency requires the couplings to be symmetric. Since in reality the supermultiplets are not degenerate, the actual couplings will be only approximately symmetric.

Now for algebras which have the property (α), one can argue that the existence of supermultiplets requires the interactions between the particles to be symmetric, at least for the low-lying states. The physics behind the arguments is as follows. First, if the bootstrap is to generate degenerate supermultiplets in a self-consistent fashion, it is evident that the interactions of the particles must have some special properties. Secondly, we know that the currents which lead to the representation matrices $f_{\alpha\alpha'}{}^i$ are themselves determined by the strong interactions. Clearly, if the interactions between multiplets are going to be such as to produce, in a self-consistent manner, the group-theoretic matrices $f_{\alpha\alpha'}{}^i$, then these interactions must again have some special properties. Now for an algebra of the type (α) we can imagine a set of symmetric couplings which would, in fact, lead to interactions with the above properties. The essential point is then, to find out if the approximately symmetrical interaction is likely to be the only interaction with these properties. If we restrict our considerations to the low-lying states where we can use simple bootstrap models as a guide, one can argue that the symmetrical interaction is, in fact, necessary.

As one means for ruling out nonsymmetrical interactions in the case (α), we cite the work of Cutkosky,[19] recently extended by Hwa and Patil,[20] which shows that, at least in the simpler sort of bootstrap models, the self-consistency of degenerate supermultiplets requires that the couplings respect a symmetry group.

One reaches the same conclusion with an argument based directly on the presumed self-consistency of the currents which make up the algebra. We will go through this argument in some detail, remaining still with the case (α). In doing so, we will use the notation $|\alpha\nu\rangle$ for the single-particle states, where ν runs over the different supermultiplets and α labels the particular members within a supermultiplet; in this notation we have

$$\langle\alpha\nu|F^i|\alpha'\nu'\rangle=(2\pi)^3\delta(\mathbf{p}_\alpha-\mathbf{p}_{\alpha'})\langle\alpha\nu|J^i(0)|\alpha'\nu'\rangle =(2\pi)^3\delta(\mathbf{p}_\alpha-\mathbf{p}_{\alpha'})f_{\alpha\alpha'}{}^{\nu,i},$$

where the $f_{\alpha\alpha'}{}^{\nu,i}$ form a representation of our algebra. Now apart from some kinematic factors which are unimportant in the present context, $f_{\alpha\alpha'}{}^{\nu i}$ is equal to the amplitude $J_{\alpha\alpha'}{}^i$ defined in Sec. II. In our self-consistent approach to the currents, $f_{\alpha\alpha'}{}^{\nu i}$ is then given by

$$f_{\alpha\alpha'}{}^{\nu i}=\sum_{cd} X_{\alpha\alpha'\nu,cd}{}^i J_{cd}{}^i, \tag{36}$$

where c and d run over all hadron states and, as before, $J_{cd}{}^i \propto \langle c|J^i(0)|d\rangle$. Now for a low-lying supermultiplet ν, it is reasonable to suppose that the right-hand side of (36) is dominated by the matrix elements of J^i between the various low-lying multiplets; we write

$$f_{\alpha\alpha'}{}^{\nu i}\approx \sum_{\alpha''\alpha'''\nu'} X_{\alpha\alpha'\nu,\alpha''\alpha'''\nu'}{}^i f_{\alpha''\alpha'''}{}^{\nu' i}. \tag{37}$$

Since we are supposed to know the f's (recall that they are representation matrices) Eq. (37) can be used to obtain some information about X, which since X depends on the strong interactions will lead to information about the couplings of the supermultiplets. To analyze the content of (37), let us suppose for simplicity that only one supermultiplet enters; suppressing the index ν we then have

$$f_{\alpha\alpha'}{}^i\approx \sum_{\alpha''\alpha'''} X_{\alpha\alpha',\alpha''\alpha'''}{}^i f_{\alpha''\alpha'''}{}^i. \tag{38}$$

Equation (38) is supposed to hold for every i; thus, if we have M currents J^i, $i=1\cdots M$ and N states, $\alpha=1\cdots N$ in our supermultiplets, (38) amounts to MN^2 equations for the matrix elements of X. There are all together MN^4 matrix elements of X so it does not appear, at first sight, that we can determine very many parameters from (38). However, in the simpler sort of bootstrap models which probably have some validity for the low-lying states, these MN^4 matrix elements of X are not all independent, but are functions of a much smaller number of coupling constants and mass ratios. For example, if we had the algebra of $SU(3)$ and had identified a supermultiplet of eight scalar mesons which bootstrap themselves and produce the eight $SU(3)$ currents self-consistently, then in the usual sort of approximation used in bootstrapping, $X_{\alpha\alpha',\alpha''\alpha'''}$ would depend only on the coupling constants $g_{\alpha'\alpha''\alpha'''}$ for the three meson vertex. Taking account of the fact that the g's must have permutation symmetry, X would be a function of at most 120 parameters but (38) would amount to $8(8)^2=512$ conditions so that Eq. (38) would, in fact, highly overdetermine the couplings. Since we know that (38) can be satisfied with symmetric couplings, the interactions of our meson octet will respect $SU(3)$. This is, of course, a very much oversimplified example, but by a similar equations-counting procedure, one can convince himself that even in more realistic cases Eq. (38) or the more generally applicable Eq. (37) will often be sufficiently restrictive to rule out interactions other than the symmetric one. In very complicated situations it may turn out that, by themselves, neither this argument nor the Cutkosky argument about self-consistency of degenerate supermultiplets, will be enough to fix the interaction, but taking these two arguments together, one expects that the odds against a nonsymmetric interaction will be very high.

So far, we have restricted ourselves to the interactions between low-lying supermultiplets where we can use simple bootstrap models as a guide. It would not however, be unreasonable to suppose that a symmetry in the interaction of low-lying states might propagate throughout the whole bootstrap, producing in the end a strong-interaction S matrix which respects the symmetry.

To recapitulate case (α), we have argued that given degenerate supermultiplets, self-consistency would generally force the couplings into a unique pattern. Then since we knew that symmetric interactions would satisfy the self-consistency conditions, we concluded that the couplings should be symmetric. Since in reality the supermultiplets are not degenerate, of course, the actual couplings will only be approximately symmetric.

Now in the case (β) it is, of course, still true that the self-consistency of the supermultiplets and currents require that the interactions have some special properties. For example, (37) still holds and the equation-counting procedure discussed above is still valid. Thus it appears that self-consistency again greatly constrains the couplings. But this time we do not have any obvious simple solution, so the nature of the constrained couplings is hard to guess. It appears that in this case we simply have to work with a symmetry that defines supermultiplets but not a symmetrical S matrix.

[19] R. Cutkosky, Phys. Rev. **131**, 1888 (1963).
[20] R. Hwa and S. Patil (to be published).

Having found that the current algebra can lead to an approximate symmetry in the bootstrap, a natural next step is to study the structure of the deviations from exact symmetry. Here it is worth commenting that, as we have shown in a previous paper,[21] the deviations from symmetry can be studied in a bootstrap framework even if, as we would expect in the case of symmetries defined by current algebras, there is no completely symmetric solution to the bootstrap from which one can start.

We conclude this section with a few remarks to the effect that it is hard to understand how there can be a general correlation between symmetries and current algebras if the strong interactions have important nonbootstrap properties. In Sec. V, we pointed out that there is no particular reason why commutators should converge rapidly, except in a bootstrap theory. Actually, one can argue that they *cannot* converge rapidly if there are important nonbootstrap terms in the currents. To see this, we suppose that the hadrons are endowed with some sort of "elementary" weak and electromagnetic currents which lead to a Lie algebra

$$[F_i,F_j]=i\sum_k C_{ijk}F_k\,. \qquad (12)$$

Now if the currents are "elementary," they are presumably specified by some set of parameters C_0 which can be chosen arbitrarily. The structure constants in (12) will depend on these parameters. On the other hand, we presented explicit evidence in (I) that the matrix elements of the currents and *a fortiori* the matrix elements of the F's taken between low-lying states are largely determined by self-consistency and will be nearly independent of the parameters C_0. Clearly, if the C_{ijk} in (12) depend on the C_0 but the matrix elements of the F's between low-lying states do not, the left-hand side of (12) is not, except by mysterious accident, going to be dominated by low-mass intermediate states. Since rapid convergence of commutators is necessary for a symmetry, it is difficult to see how a general correlation between current algebras and symmetry could arise in a theory containing essential nonbootstrap elements.

VII. $SU(3)$ AND $SU(6)$

In this section, we wish to see how $SU(3)$ and $SU(6)$ can be understood in terms of our discussion of current algebras.

We will assume to start with that the observed weak and electromagnetic currents generate[1,22] the algebras of $SU(3)$ and $SU(6)$. Ideally we would like to be able to derive these algebras from bootstrap relations; later in this section we discuss to what extent this is possible in simple models.

Having assumed the algebras of $SU(3)$ and $SU(6)$, we can use the machinery developed in Secs. V and VI to discuss the connection between these algebras and strong interaction symmetries. In doing so, we will use the notation $F^{i0}=\int J^{0i}d^3x$ for the space integrals of the time components of the eight vector currents, and $F_5{}^{iz}=\int J_5{}^{zi}d^3x$ and similarly defined $F_5{}^{ix}$ and $F_5{}^{iy}$ for the integrals of the space components of the axial vector currents. The F^{i0} are then supposed to generate $SU(3)$, and adding the $F_5{}^{ix}$, $F_5{}^{iy}$ and $F_5{}^{iz}$ enlarges the algebra to $SU(6)$. For future reference, we note that the operators F^{i0} and $F_5{}^{iz}$ generate $SU(3)\otimes SU(3)$.

We begin our discussion with $SU(3)$. Let us suppose here we have not yet discovered that the strong interactions are nearly $SU(3)$ symmetric. What could we say, from the general principles of Secs. V and VI, about the prospects for finding a symmetry? First we know that since the F^{i0}'s are integrals of time components of currents, the commutators should converge rapidly in either of the cases $\mathbf{p}=0$ or $\mathbf{p}=\infty$. One needs, then, only to know that single-particle states dominate in the commutator. This will be virtually automatic for $\mathbf{p}=\infty$. To see that it should also hold for $\mathbf{p}=0$, we recall that single-particle states will dominate the commutators for $\mathbf{p}=0$ if whatever resonances appear in the form factors of the divergences of the currents have masses large compared to the spacing between neighboring single-particle states. Since there are no well-established low-mass candidates[23] for resonances in these form factors, single-particle states should dominate. On these grounds we would expect, according to Sec. VI, to find supermultiplets associated with the current algebra of $SU(3)$. Finally, since $SU(3)$ is an algebra of the type (α), there should be a corresponding symmetry in interactions of the supermultiplets. Thus we would expect to find a rather complete $SU(3)$ symmetry in the strong interactions, as is, of course, the experimental situation.

Let us now consider the enlarged algebra $SU(6)$. In Sec. V, it was pointed out that for $\mathbf{p}=0$ one cannot expect commutators of objects like F^{iz} which involve integrals of space components of currents to be dominated by single-particle intermediate states. To obtain dominance by single-particle intermediate states we must, in this case, take $\mathbf{p}=\infty$. However, if we take $\mathbf{p}\rightarrow\infty$ along the z axis, commutators involving $F_5{}^{ix}$ and $F_5{}^{iy}$ will not, according to Sec. V, converge rapidly. We conclude, then, that only the subalgebra $SU(3)\otimes SU(3)$ generated by F^{i0} and $F_5{}^{iz}$ will lead to supermultiplets. Thus, it appears that the complete algebra $SU(6)$ is not likely to be of use in generating supermultiplets, but that we must work with the smaller algebra $SU(3)\otimes SU(3)$ and supermultiplets defined with $\mathbf{p}=\infty$. Fortunately, the reduced algebra will lead to many of the same results as $SU(6)$. For example, let us consider the baryons B and resonances B^* which are usually placed in the **56** representation of $SU(6)$. The generators F^{i0} and $F_5{}^{iz}$ of our $SU(3)\otimes SU(3)$ clearly can connect states with different spin, but do

[21] R. Dashen and S. Frautschi, Phys. Rev. (to be published).

[22] R. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters **13**, 678 (1964).

[23] Since only the strangeness-changing currents have a divergence, only strange 0^+ mesons are candidates.

preserve helicity. We can try, then, to put the baryons and resonances with helicity $\frac{1}{2}$ into one "supermultiplet" and the resonances with helicity $\frac{3}{2}$ into another supermultiplet. There will be a consistency condition: The helicity $\frac{1}{2}$ "supermultiplet" will determine the B^*-B^* matrix elements of the F's which must then agree with the corresponding quantities obtained from the helicity $\frac{3}{2}$ "supermultiplet." This condition will be satisfied if for helicity $\frac{1}{2}$ we put the baryons and resonances into $(\mathbf{6},\mathbf{3})$, which reduces to $\mathbf{8}+\mathbf{10}$ in $SU(3)$, and for helicity $\frac{3}{2}$ we put the resonances in $(\mathbf{10},\mathbf{1})$ which reduces to $\mathbf{10}$ in $SU(3)$. One can verify that this assignment of B and B^* to $SU(3)\otimes SU(3)$ supermultiplets reproduces the usual static $SU(6)$ predictions for the B-B, B-B^*, and B^*-B^* matrix elements of the vector and axial currents.[24]

Having shown how the $SU(3)$ and $SU(6)$ current algebras can lead to supermultiplets containing the 56 baryons, let us go back now and see to what extent one could have derived these current algebras and obtained a fully self-consistent picture.

In Secs. III and IV we showed that self-consistent currents lead to current algebras, even if the currents are not conserved. Thus the remaining problems are to demonstrate the existence of self-consistent currents, and to show that these currents specifically generate the algebras of $SU(3)$ and $SU(6)$.

In the first paper of this series, we studied the existence of self-consistent currents in a reciprocal bootstrap model. In order to generate $SU(6)$ we need eight self-consistent vector currents and nine self-consistent axial currents. Our bootstrap model gave the eight vector currents and eight of the axial currents. The missing axial current in an $SU(3)$ singlet which does not couple B to B^* and, in $SU(6)$, has much larger B^*-B^* than B-B matrix elements. The existence of a self-consistent current with these properties is not really incompatible with the model, since our model takes account of B-B and B-B^* but not B^*-B^* matrix elements of currents. We also obtained definite predictions for ratios among the B-B and B-B^* matrix elements of the currents. These predictions were numerically very close to what we find by assuming that B and B^* belong to supermultiplets associated with an $SU(6)$ current algebra.

From the results of the approximate calculation described above, it would not seem unreasonable that there may be self-consistent currents with the quantum numbers necessary to generate $SU(6)$. To see if the currents actually generate $SU(6)$ we would need enough of their matrix elements to construct the commutators. Since we have only the B-B and B-B^* matrix, the best we can do is to take commutators sandwiched between two B states and assume that it is dominated by B and B^* intermediate states. If we restrict ourselves to the $SU(3)\otimes SU(3)$ algebra discussed above, this approximation should be reasonably good. Upon doing this we find that our bootstrap is compatible only with the currents generating $SU(6)$.

In conclusion, we would like to point out that there are definite advantages in looking at both current algebras and dynamical models at the same time. We have two specific examples in mind:

(i) If we knew only that we had the $SU(3)\otimes SU(3)$ algebra and that there were 56 low-lying baryons, the $(\mathbf{6},\mathbf{3})$-$(\mathbf{10},\mathbf{1})$ assignment described above would not be unique. We could, for example, assign the baryons to $(\mathbf{8},\mathbf{1})$ and the resonances to $(\mathbf{10},\mathbf{1})$. The latter assignment would, however, require that the B-B^* matrix elements of the currents vanish which is incompatible with self-consistency in the dynamical model of the previous paper.

(ii) From a purely algebraic point of view, many quantities like baryon magnetic moments and B-B matrix elements of the axial vector currents appear to be independent. However, in a static bootstrap model the self-consistency requirements on these quantities are, as discussed in the previous paper, exactly the same. The dynamics tells us, then, that independently of any symmetry consideration the pattern of B-B magnetic moments will be the same as the pattern of couplings to the axial current. From this one concludes that if a current algebra like our $SU(3)\otimes SU(3)$ gives the correct matrix elements of the axial currents between baryons, it will automatically give the correct magnetic moments.

---

[24] Strictly speaking, this is only true if mass differences are neglected. The present $SU(3)\otimes SU(3)$ says that the matrix elements like

$$f_{5\alpha\alpha'}{}^{iz}(p_z\rightarrow\infty)=\lim_{p_z\rightarrow\infty}\langle\alpha|J_5{}^{iz}(0)|\alpha'\rangle$$

are approximately Clebsch-Gordan coefficients. On the other hand, static $SU(6)$ with $\mathbf{p}=0$ predict Clebsch-Gordan coefficients for

$$f_{5\alpha\alpha'}{}^{iz}(\mathbf{p}=0).$$

If mass-difference effects are included, these predictions may differ (Ref. 13).